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MODES WITH SWITCHINGS OF INCREASING FREQUENCY IN THE PROBLEM OF CONTROLLING A ROBOT*

V.F. BORISOV and M.I. ZELIKIN

Trajectories that are optimal with respect to high-speed response are constructed for a system for controlling a two-component manipulator (a robot). It is shown that when the initial conditions lie within a certain open region of the phase space, all optimal trajectories will have a segment of switchings of increasing frequency (SIF), i.e. a segment in which the control will undergo an infinite number of switchings in a finite time interval.

The synthesis of the optimal control in the R^2 plane containing the mode of SIF was first constructed by Fuller /1/. It was shown in /2/ that the synthesis is structurally stable in the sense that adding terms of higher order of smallness to the integrand and to the right-hand sides of the system of differential constraints does not affect the qualitative pattern of the optimal synthesis in the neighbourhood of the origin of coordinates.

The present paper explains that the synthesis in the problem of optimal control (relative to the high speed response) of the motion of the robot appears, in a certain sense, a direct product of the synthesis appearing in the Fuller problem and of the synthesis in the simplest problem of high-speed response (/3/, pp.38-47). The special aspect of the present paper consists of the proof of the proposition that switching surface is a piecewise-smooth manifold. The presence of the SIF mode is connected only with the fact that every trajectory intersects this surface an infinite number of times. In existing papers, the piecewise smoothness of the switching curve was proved for the two-dimensional problems using the SIF mode only for problems admitting of a one-parameter group of symmetries /1, 4-6/. A proof of the presence of SIF was given in /7, 8/.

1. Formulation of the problem. The problem of controlling the robot can be formulated in two ways /9/. A movable element is fixed on a massive vertical cylinder rotating about its axis. In the first version the movable element has the form of a bar rotating in the vertical plane, and in the second version it takes the form of a horizontal advancing arrow. The system has two control parameters, the moment acting on the vertical cylinder, and

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the force acting on the movable element. Both controls are constrained in modulo.

The equations of motion for the first version have the form

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = (v - x_2 x_4 \sin 2x_3)/(1 + \sin^2 x_3)$$
$$x_3^{\cdot} = x_4, \quad x_4^{\cdot} = u + \sin x_3 + C x_2^2 \sin 2x_3 \tag{1.1}$$

and for the second version we have

$$x_1^{\cdot} = x_2/(1 + x_3^2), \quad x_2^{\cdot} = v, \quad x_3^{\cdot} = x_4$$
$$x_4^{\cdot} = u + x_2^2 x_3/(1 + x_3^2)^2 \tag{1.2}$$

The controls $u(\cdot), v(\cdot)$ are measurable functions, and

$$|u| \leqslant u_0, \quad |v| \leqslant 1 \tag{1.3}$$

The trajectories $X(\cdot) = (x_1(\cdot), x_2(\cdot), x_3(\cdot), x_4(\cdot))$ are absolutely continuous.

We shall consider the problem (Problem A1)

$$T \to \min, \quad X(0) = X_0, \quad X(T) = 0$$

where the trajectory $X(t)$ satisfies system (1.2), (1.3). All results obtained remain valid also for the problem in which the trajectories $X(t)$ satisfy system (1.1), (1.3).

Definition. We shall call the trajectory $X(t)$ with control $u(t), v(t)$ the mode of switching of increasing frequency (SIF), if the control has an infinite number of discontinuities in a finite time interval.

Below we shall show that for any A, B: $B < 0$, $A > {}^1/_2 B^2$ with initial conditions $X_0 \in D_\varepsilon = \{X \mid \max(\sqrt{|x_4|}, |x_3|, |x_1 - A|, |x_2 - B|) < \varepsilon\}$, the optimal trajectories (OT) of Problem A1 contain, for sufficiently small ε, the segments of SIF. After a finite time interval depending on the initial conditions in a continuous manner, the OT with an infinite number of switching of the component of the control u and with constant $v = -1$, will emerge onto the singular manifold of the Problem A1 represented by the surface $\pi = \{X \mid x_3 = x_4 = 0\}$. After this, v will arrive, with $u = 0$ and a single switching, at the origin of coordinates. A one-parameter family of OT filling the two-dimensional surface $\Sigma_{\alpha,\beta}$, smooth outside $\Sigma_{\alpha,\beta} \cap \pi$, arrives at every point $(\alpha, \beta, 0, 0) \in D_\varepsilon$. The region D_ε decomposes into the layers $\Sigma_{\alpha,\beta}$ over the base π. The points of switching the OT form a three-dimensional surface P (of class C^1 outside $P \cap \pi$), dividing the region D_ε into two subregions D_ε^+ and D_ε^- in which $u = 1$ and $u = -1$ respectively.

2. Reduction of the problem.

Let us consider the following system of equations:

$$x_1^{\cdot} = \sigma x_2, \quad x_2^{\cdot} = v, \quad x_3^{\cdot} = x_4$$
$$x_4^{\cdot} = u + x_2^2 x_3/(1 + x_3^2)^2 \tag{2.1}$$

where the control $\sigma \in [0, 1], u, v,$ and conditions (1.3) all hold. Let us denote by M_R the set of absolutely continuous trajectories $X(\cdot), X(0) = X_0$ satisfying system (1.2) at $0 \leqslant t \leqslant \varepsilon R$, and system (2.1) at $t > \varepsilon R$. The constant R will be chosen later.

Let us consider the following problem (Problem A2):

$$T \to \inf, \quad X(0) = X_0, \quad X(T) = 0, \quad X(\cdot) \in M_R$$

By virtue of Filippov's theorem /10/ Problems A1 and A2 have solutions for any initial conditions.

Note 1. The optimal time in Problem A2 does not exceed the optimal time in Problem A1. Therefore, if the OT of Problem A2 is found to be admissible in the Problem A1, then it will also represent the OT in A1. Below we shall show that Problem A1 and A2 are indeed equivalent.

All subsequent assertions will be valid when $\varepsilon \leqslant \varepsilon_0$ for a suffficiently small $\varepsilon_0 = \varepsilon_0(A, B) > 0$, and we shall not discuss it in any detail.

Lemma 1. Let X^* be the OT in Problem A2 with initial conditions $X_0 \in D_\varepsilon$, and σ^*, u^*, v^* be the control on it. Then $\sigma^* \equiv 1$ when $t > \varepsilon R$; $v^* \equiv -1$ for $0 \leqslant t \leqslant \varepsilon R$.

Proof. When $t > \varepsilon R$, the motion in the x_1, x_2 plane does not depend on the behaviour of the coordinates x_3, x_4. When $X_0 \in D_\varepsilon$, we can move the projection of the trajectory $X(\cdot)$ on the x_3, x_4 plane to the origin of coordinates, in a time of the order of ε. Therefore the optimal time T^* is determined by the conditions $x_1(T^*) = x_2(T^*) = 0$ only.

Let us write $x_1^*(\varepsilon R) = \alpha$, $x_2^*(\varepsilon R) = \beta$. Considering just the first two equations of system (2.1), we can show that $x_2^*(t) \leqslant 0$ for any $t \geqslant 0, \sigma^* \equiv 1$ and

$$T^* = \beta + 2\sqrt{\alpha + {}^1/_2\beta^2} + \varepsilon R \tag{2.2}$$

Thus Problem A2 is reduced to that of minimizing the function (2.2) on the solutions of system (1.2), (1.3) determined for $0 \leqslant t \leqslant \varepsilon R$.

We shall show that if $v^* \neq -1$ on the set of positive measure for $0 \leqslant t \leqslant \varepsilon R$, then the trajectory X^* will not be optimal. Let us consider the trajectory X° of the system (1.2) with control $u^\circ = u^*$, $v^\circ = -1$. We have

$$z(t) = \int_0^t (1 + v^*(\tau))\, d\tau \geqslant 0 \tag{2.3}$$

$$y^{\cdot\cdot} = (x_2^*)^2 x_3^* /(1 + (x_3^*)^2)^2 - (x_2^\circ)^2 x_3^\circ/(1 + (x_3^\circ)^2)^2 \tag{2.4}$$
$$(y(t) = x_3^*(t) - x_3^\circ(t), \quad z(t) = x_2^*(t) - x_2^\circ(t))$$

Let us integrate the terms on the right-hand side of (2.4), separating $y(t), z(t)$ as factors and denoting by $a(X^*, X^\circ)$, $b(X^*, X^\circ)$ the coefficients of $y(t), z(t)$.

$$y^{\cdot\cdot} = ay + bz$$

Taking into account the fact that $y(0) = 0$, $y^\cdot(0) = 0$, we obtain

$$y(\cdot)|_{[0,\, t]} = K_t^{-1}(Z(\cdot)), \quad Z(t) = \int_0^t (t - \tau)\, b(\tau)\, z(\tau)\, d\tau$$

where we have introduced the operator

$$K_t\colon C[0, t] \to C[0, t], \quad 0 < t \leqslant \varepsilon R$$
$$(K_t w)(t) = w(t) - \int_0^t (t - \tau)\, a(\tau)\, w(\tau)\, d\tau$$

which has a bounded inverse.

From this we find ($C > 0$) that

$$|y(t)| < C\varepsilon \|z(\cdot)\|_{C[0,\, t]} = C\varepsilon z(t) \tag{2.5}$$

Let us now obtain an estimate for

$$\Delta x_1 = x_1^*(\varepsilon R) - x_1^\circ(\varepsilon R) = \int_0^{\varepsilon R} \left(\frac{x_2^*}{1 + x_3^{*2}} - \frac{x_2^\circ}{1 + x_3^{\circ 2}}\right) dt$$

Separating in the integrand the terms containing $y(t), z(t)$, and taking into account (2.5) and the fact that $|x_3^\circ(t)| < O(\varepsilon)$, we obtain

$$\Delta x_1 \geqslant (1 - O(\varepsilon)) \int_0^{\varepsilon R} z(t)\, dt$$

Now from (2.2) and (2.3) we obtain the inequality $T^*(X^\circ(\cdot)) < T^*(X^*(\cdot))$, which is impossible.

From Lemma 1 and the identity

$$(t + x_2 + 2\sqrt{x_1 + {}^1\!/_2 x_2^2})\big|_{t=0}^{t=\varepsilon R} = I$$
$$I = \int_0^{\varepsilon R} F(X)\, dt, \quad F(X) = -x_2 x_3^2 (x_1 + {}^1\!/_2 x_2^2)^{-1/2} (1 + x_3^2)^{-1}$$

satisfied on the solutions of the system $x_1^\cdot = x_2/(1 + x_3^2)$, $x_2^\cdot = -1$, it follows that Problem A2 is equivalent to the problem

$$I \to \inf; \quad x_1^\cdot = x_2/(1 + x_3^2), \quad x_2^\cdot = -1, \quad x_3^\cdot = x_4$$
$$x_4^\cdot = u + x_2^2 x_3/(1 + x_3^2)^2, \quad X(0) = X_0, \quad |u| \leqslant u_0$$

3. Mode of switchings at increasing frequency.

Theorem 1. The solutions X^* of Problem A2 with initial conditions $X_0 \in D_\varepsilon$, have a segment of SIF. The trajectories X^* emerge, after a finite time shorter than εR, and depending continuously on X_0 onto the singular mode $x_3 = x_4 = 0$ with an infinite number of switchings of the control u. The optimal motion is completed by a segment of the singular trajectory with $u = 0$.

Proof. 1°. The upper bound of the optimal value of the functional in Problem A2. When $0 \leqslant t \leqslant \varepsilon R$, we have, for any admissible trajectory of Problem A2, the inclusion $X(t) \in D_{C_1\varepsilon}$. Therefore

$$\max_{X(\cdot) \in M_R} (x_2^2 x_3/(1 + x_3^2)^2\colon 0 \leqslant t \leqslant \varepsilon R) \leqslant \gamma_0 < C_2 \varepsilon$$

Let $x^\circ = (x_3^\circ, x_4^\circ)$ be the OT in the problem

$$T \to \inf;\quad x_3^{\cdot} = x_4,\quad x_4^{\cdot} = u,\quad |u| \leqslant u_1 = u_0 - \gamma_0$$
$$x_i(0) = x_{i0},\quad x_i(T) = 0,\quad i = 3, 4$$

We shall consider the set $K(r) = \{x_3, x_4 \mid |x_3| + (2u_1)^{-1}x_4^2 \leqslant r\}$. Then, if $X_0 \in D_\varepsilon$, then $(x_{30}, x_{40}) \in K(\lambda\varepsilon^2)$, $\lambda = 1 + (2u_1)^{-1}$. Let $(x_{30}, x_{40}) \in K(r)$, $r \leqslant \lambda\varepsilon^2$. The time τ° of arrival of $x^\circ(\cdot)$ at the point $(0, 0)$ has the upper bound $C_3\sqrt{r}$. Moreover, $x^\circ(t) \in K(r)$ for $t \in [0, \tau^\circ]$. Let us denote by $X^\circ(\cdot)$ the admissible trajectory in Problem A2, whose projection on the space x_3, x_4 coincides with $x^\circ(\cdot)$. Then

$$\inf_{X(\cdot)\in M_R} I \leqslant (r^2 + O(\varepsilon))(A + {}^1/_2 B)^{-1/2}\int_0^{\tau^\circ}(-x_2^\circ)\,dt \leqslant sr^{5/2} \tag{3.1}$$

for certain $s > 0$, it is possible to select a value independent of R.

2^O. *The lower bound of the optimal value of the functional in Problem A2.* In the region $\Omega_1 = \{x_3, x_4 \mid |x_3| \leqslant {}^1/_2 r\} \setminus K(r)$ the component $x_3^{\cdot} = x_4$ of the phase velocity is separated from zero: $|x_3^{\cdot}| \geqslant \sqrt{ru_1}$, therefore the time for which any admissible trajectory remains in the region Ω_1 is estimated as follows:

$$\max_{X(\cdot)\in M_R}(\tau\colon (x_3, x_4)(t) \in \Omega_1, t \in [0, \tau]) \leqslant \sqrt{r/u_1} < {}^5/_4\sqrt{r/u_0}$$

Let $X(\cdot)$ be any trajectory admissible in Problem A2 and such, that $x_3(0) = {}^1/_2 r \operatorname{sgn} x_4(0)$. Then

$$\inf_{X(\cdot)\in M_R}(\tau > 0\colon (x_3, x_4)(\tau) \notin \Omega_2) \geqslant 2\sqrt{r/u_1}/(u_0 + \gamma_0) > \tag{3.2}$$
$${}^3/_2\sqrt{r/u_0},\quad \Omega_2 = \{x_3, x_4 \mid |x_3| \geqslant {}^1/_2 r\}\setminus K(r)$$

Let X^* be the OT of the Problem A2, $X^*(0) \in D_\varepsilon$, $(x_3^*, x_4^*)(0) \in K(r)$. Let us estimate the time τ in which X^* arrives at the region $K({}^1/_2 r)$, i.e. let us assume that $(x_3^*, x_4^*)(t) \notin K({}^1/_2 r)$ when $t \in [0, \tau]$ $(\tau \leqslant \varepsilon R)$. Let n be the number of intersections of the trajectory X^* with the planes $x_3 = {}^1/_2 r \operatorname{sgn} x_4$ at $t \in [0, \tau]$. We write $\mu = \{t \in [0, \tau] \mid |x_3^*(t)| \leqslant {}^1/_2 r\}$, $\nu = [0, \tau] \setminus \mu$. By virtue of (3.2) we have $\tau > {}^3/_2(n - 1)\sqrt{r/u_0}$, therefore

$$n < 1 + {}^2/_3\tau\sqrt{u_0/r},\quad \operatorname{mes}\mu < {}^5/_4 n\sqrt{r/u_0} \leqslant {}^5/_4\sqrt{r/u_0} + {}^5/_6\tau,$$
$$\operatorname{mes}\nu \geqslant {}^1/_6\tau - {}^5/_4\sqrt{r/u_0}$$

By virtue of (3.1) we have

$$\int_0^\tau F^*\,dt \leqslant sr^{5/2},\quad F^* = F(X^*) \tag{3.3}$$

On the other hand we find, that

$$\int_0^\tau F^*\,dt \geqslant \int_\nu F^*\,dt \geqslant Q_1 r^2 \operatorname{mes}\nu \geqslant Q_2 r^2\tau - Q_3 r^{5/2} \tag{3.4}$$

for some $Q_i > 0$, independent of R. From (3.3) and (3.4) it follows that $\tau < Q\sqrt{r}$.

Thus after a period not exceeding $Q\sqrt{r}$, the value of r is halved, after the time of $Q\sqrt{{}^1/_2 r}$ it halves again, and so on. It follows therefore that when $r < \lambda\varepsilon^2$, then the time in which the OT X^* with initial conditions $X_0 \in D_\varepsilon$ arrives at the plane $x_3 = x_4 = 0$, has an upper bound in the form of the following sum of a geometric progression:

$$\tau < Q(\sqrt{r} + \sqrt{2^{-1}r} + \ldots + \sqrt{2^{-n}r} + \ldots) = \frac{Q\sqrt{r}}{1 - 2^{-1/2}} \leqslant Q^*\varepsilon$$

Let us select $R > Q^*$. Then $\tau < \varepsilon R$ and by virtue of Note 1 the OT in Problem A2 will be the OT in Problem A1. Thus we have shown that the OT X^* of Problem A1 arrives at the plane π in a time shorter than εR, which is continuously dependent on initial conditions $X_0 \in D_\varepsilon$.

3^O. *Order of the singular OT's.* We shall show that the OT of Problem A1 with $X_0 \in D_\varepsilon$, have segments of SIF. To do this it is sufficient to show that the OT of Problem A1, singular in it and belonging to D_ε, are of even order (see /11/ for the definition of the order of a singular trajectory).

Let us consider the system of equations of the Pontryagin maximum principle /12/ for Problem A1

$$\begin{gathered}\psi_1^{\cdot} = 0, \quad \psi_2^{\cdot} = -\psi_1/(1+x_3^2) - 2x_2x_3\psi_4/(1+x_3^2)^2 \\ \psi_3^{\cdot} = 2\psi_1x_2x_3/(1+x_3^2)^2 - x_2^2(1-3x_3^2)\psi_4/(1+x_3^2)^3 \\ \psi_4^{\cdot} = -\psi_3, \quad x_1^{\cdot} = x_2/(1+x_3^2), \quad x_2^{\cdot} = v^* \\ x_3^{\cdot} = x_4, \quad x_4^{\cdot} = u^* + x_2^2x_3/(1+x_3^2)^2 \\ \max_{|u|\leqslant u_0} u\psi_4(t) = u^*(t)\psi_4(t), \quad \max_{|v|\leqslant 1} v\psi_2(t) = v^*(t)\psi_2(t)\end{gathered} \tag{3.5}$$

Let us determine the trajectories singular in u, lying in D_ε. Let $\psi_4(t) = 0$ on some segment (τ_0, τ_1). Then $\psi_3(t) = 0$. Differentiating the identity $\psi_3 = 0$ we obtain, by virtue of the system of Eqs.(3.5), $x_2x_3\psi_1 = 0$. From Lemma 1 it follows that $x_2 \neq 0$ in D_ε. If $\psi_1 = 0$, then $\psi_2 = \text{const} \neq 0$ and the optimal control u^* will constantly violate Lemma 1. Therefore $\varphi_1 \neq 0$, and we have $x_3 = x_4 = u^* = 0$. Differentiating the relations $H_1 = \psi_4$ we find, by virtue of system (3.5), that at the segment of singular control

$$\frac{\partial}{\partial u}\frac{d^kH_1}{dt^k} = 0, \quad k = 0, 1, 2, 3; \quad \frac{\partial}{\partial u}\frac{d^4H_1}{dt^4} = 2x_2\psi_1 \neq 0$$

Therefore, the order of the singular extremal is zero. The validity of the initial assertion now follows from the fact that the singular OT of even order cannot combine with the piecewise smooth non-singular OT if the control is discontinuous at the point of conjugation /11/. This completes the proof of Theorem 1.

4. Smoothness of the switching surface. *Theorem 2.* Problem A2 has a two-parameter set of families of extremals $N_{\alpha,\beta}$ of the following type. For fixed values of α, β the trajectories of the family $N_{\alpha,\beta}$ pass through the point $X_{\alpha,\beta} = (\alpha, \beta, 0, 0) \in D_\varepsilon$ and contain the segment of SIF. The points of switching of the trajectories $N_{\alpha,\beta}$ form, for fixed $X_{\alpha,\beta}$, a one-dimensional curve smooth outside π. The set P of points of switching for all trajectories $N_{\alpha,\beta}$ with $X_{\alpha,\beta} \in D_\varepsilon$, is a three-dimensional surface smooth everywhere except, perhaps, at the points $P \cap \pi$.

Proof. Let us put, for brevity, $u_0 = 1$. We shall seek a family $N_{\alpha,\beta}$ of trajectories of system (3.5) passing through the point

$$\begin{gathered}x_1 = \alpha, \quad x_2 = \beta, \quad x_3 = x_4 = \psi_3 = \psi_4 = 0 \\ \psi_1 = -1, \quad \psi_2 = -\beta - \sqrt{\alpha + {}^1/_2\beta^2}\end{gathered}$$

Let us consider the mapping of the continuation Φ of the surface $\Sigma = \{\xi = (\Psi, X) \mid \psi_4 = 0\}$ onto itself, forming together with the point $\xi_0 \in \Sigma$, the point $\xi_1 = \Phi\xi_0 \in \Sigma$ of intersection of the trajectory $\xi(t, \xi_0)$, $\xi(0, \xi_0) = \xi_0$ of system (3.5) with the surface Σ, at negative t smallest in modulo. We denote by $\tau(\xi_0)$ the negative root of the equation $\psi_4(\tau, \xi_0) = 0$ smallest in modulo.

We replace the variables $\xi = (\Psi, X)$ by the variables $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$, using the formulas

$$\begin{gathered}x_1 = \alpha + u\lambda_1\lambda_3, \quad x_2 = \beta + u\lambda_1\lambda_4 \\ x_3 = -u\lambda_1^2\lambda_2, \quad x_4 = \lambda_1, \quad \psi_2 = -\beta - \sqrt{\alpha + {}^1/_2\beta^2} + u\lambda_1\lambda_6, \\ \psi_3 = \lambda_1^3\lambda_5, \quad u = \operatorname{sgn}\lambda_1\end{gathered} \tag{4.1}$$

The above formulas specify one of the versions of determining the singularity of the mapping Φ: instead of the point $x_1 = \alpha$, $x_2 = \beta$, $x_3 = 0$, $x_4 = 0$, $\psi_2 = -\beta - \sqrt{\alpha + {}^1/_2\beta^2}$, $\psi_3 = 0$, $\psi_4 = 0$ we insert the plane $\lambda_1 = 0$.

We shall seek a switching curve on the solutions of the family $N_{\alpha,\beta}$, starting from the fact that the curve must be an invariant curve of the mapping Φ. We rewrite Eq.(3.5) in the form of integral equations, and make the substitution (4.1). We shall seek a solution of the equation $\psi_4(\tau, \xi_0) = 0$ in the form $\tau = u\lambda_1 s$. We shall agree to denote by $O_k(\varepsilon)$ the function Λ such, that $\| O_k(\varepsilon)\|_{C^k(\Omega)} \to 0$ as $\varepsilon \to 0$ (where Ω is an arbitrary compact region of R^6 corresponding to D_ε under the substitution (4.1)). Let $\Lambda^1 = \Phi(\Lambda)$. Then we obtain

$$\begin{gathered}\lambda_1^1 = (1+s)\lambda_1 + O_1(\varepsilon), \quad \lambda_2^1 = (-\lambda_2 + s + {}^1/_2 s^2) \times \\ (1+s)^{-2} + O_1(\varepsilon), \quad \lambda_3^1 = -(\lambda_3 + \beta s + u\lambda_1(\lambda_4 s - \\ {}^1/_2 s^2))(1+s)^{-1} + O_1(\varepsilon), \quad \lambda_4^1 = (-\lambda_4 + s)(1+s)^{-1} +\end{gathered} \tag{4.2}$$

$$O_1(\varepsilon),\quad \lambda_5^1 = (\lambda_5 + \beta(2\lambda_2 s - s^2 - {}^1/_3 s^3) + u\lambda_1(2\lambda_2\lambda_4 s - (\lambda_2 + \lambda_4)s^2 - {}^1/_3(2 - \lambda_4)s^3 + {}^1/_4 s^4))(1+s)^{-3} + O_1(\varepsilon),\quad \lambda_6^1 = -(\lambda_6 + s)(1+s)^{-1} + O_1(\varepsilon)$$

We obtain the following equation for $s = s(\Lambda)$:

$$0 = -\lambda_5 + \beta(-\lambda_2 s + {}^1/_3 s^2 + {}^1/_{12} s^3) + u\lambda_1(-\lambda_2\lambda_4 s + {}^1/_3(\lambda_2 + \lambda_4)s^2 + {}^1/_{12}(\lambda_4 - 2)s^3 - {}^1/_{20}s^4) + O_1(\varepsilon) \tag{4.3}$$

Direct calculation confirms that the bound of the mapping $\Phi|_{\lambda_1=0}$ has a unique fixed point $\Lambda^\circ = \Lambda^\circ(\alpha, \beta)$ with $\lambda_2^\circ > 0$.

Let us fix α, β: $|\alpha - A| < \varepsilon$, $|\beta - B| < \varepsilon$. For any sufficiently small neighbourhood U_0 of the point Λ°, the constraints $\Phi^2|_{U_0 \cap \{\Lambda:\lambda_1>0\}}$ and $\Phi^2|_{U_0\cap\{\Lambda:\lambda_1<0\}}$ (where $\Phi^2(\Lambda) = \Phi(\Phi(\Lambda))$) are obviously continued to the diffeomorphism onto the whole neighbourhood U_0. We shall denote these continuations by Φ_+^2 and Φ_-^2. We can directly confirm that they have a saddle point structure at the fixed point $\Lambda^\circ(\alpha, \beta)$: the Jacobians $D\Phi_\pm^2(\Lambda^\circ(\alpha,\beta))$ have each a single eigenvalue greater than unity with the eigenvectors $\varphi_\pm$, and all remaining eigenvalues are less than unity in modulo. According to the theorem on invariant manifolds for a difeomorphism /13/, the mapping Φ has a one-dimensional unstable invariant manifold $\Gamma_{\alpha,\beta}$, smooth outside the point $\lambda_1 = 0$, and tangent to the eigenvectors φ_+ when $\lambda_1 > 0$, and to φ_- when $\lambda_1 < 0$. It can be shown that the set $\Gamma = \bigcup_{\alpha,\beta} \Gamma_{\alpha,\beta}$ is a three-dimensional manifold of the form $\Lambda = f(\lambda_1, \alpha, \beta)$ of class C^1 outside the plane $\lambda_1 = 0$.

Let us denote by $N_{\alpha,\beta}$ the family of trajectories of system (3.5) with initial conditions corresponding, according to the formulas (4.1), to the points of the manifold Γ. $N_{\alpha,\beta}$ has all the properties demanded in the formulation of the theorem, and this completes the proof of Theorem 2.

Lemma 2. Let $\lambda^*(\lambda, \alpha, \beta)$ be a component λ_1 of the vector $\Phi^{-1}(\Lambda)$, $\Lambda = f(\lambda, \alpha, \beta)$. Then

$$|\partial\lambda^*/\partial\lambda| < {}^1/_4,\quad |\partial\lambda^*/\partial\alpha| = O(\varepsilon),\quad |\partial\lambda^*/\partial\beta| = O(\varepsilon)$$

We prove the lemma by differentiating the mapping (4.2), (4.3) at the point $\Lambda^\circ(\alpha, \beta)$.

5. Proof of the optimality of the family of extremals. *Theorem 3*. The trajectories of the family $N_{\alpha,\beta}$ are optimal in Problem A1.

Proof. We shall use the notation of Theorem 2. We see that on the extremals constructed we have $H = \sqrt{\alpha + {}^1/_2\beta^2} \neq 0$. We shall show that the field of conjugated variables $\Psi/H = (\psi_1/H, \ldots, \psi_4/H)$ corresponding to $N_{\alpha,\beta}$ is potential in the region D_ε.

According to the theorem on the continuous dependence of the solutions of differential equations on the initial date, the function Ψ/H is continuous in D_ε. Let P be the surface of switching of the trajectories $N_{\alpha,\beta}$. We shall consider an arbitrary closed contour $\gamma^\circ \subset P \setminus \pi$, and show that

$$\oint_{\gamma^\circ} \frac{\psi}{H}\, dX = 0 \tag{5.1}$$

We denote by Φ^* the mapping of the continuation $\Phi^*: P \to P$, induced by the mapping Φ^{-1}. According to the Poincaré-Cartan theorem on integral invariant /14/, we have

$$\oint_{\gamma^n} \frac{\psi}{H}\, dX = \oint_{\gamma^\circ} \frac{\psi}{H}\, dX, \quad \gamma^n = \Phi^*\gamma^{n-1}$$

Let us choose, as the system of coordinates on P, the values $(\lambda_1, \alpha, \beta)$. We have in these coordinates $\Phi^*(\lambda_1, \alpha, \beta) = (\lambda^*(\lambda_1, \alpha, \beta), \alpha, \beta)$.

Let us write $\mathrm{pr}(\lambda_1, \alpha, \beta) = (0, \alpha, \beta)$. It is clear that it is sufficient to consider only those γ° for which $\mathrm{pr}\,\gamma^\circ$ is a piecewise smooth curve. According to Lemma 2, $\gamma^n \to \mathrm{pr}\,\gamma^\circ$ in the metric of C^1 as $n \to \infty$. The bound $\Psi/H|_\pi$ coincides with the corresponding field of conjugate variables in the simplest problem of high speed response in the π plane. We know that this field is potential, therefore the relation (5.1) holds. This, together with the fact that Ψ/H is continuous, implies that Ψ/H is potential in D_ε.

Let now X^* be the OT in Problem A1, $X^*(0) \in D_\varepsilon$ and X° be a trajectory of the family $N_{\alpha,\beta}$ with $X^\circ(0) = X^*(0)$. Consider the contour in $D_\varepsilon \cup \pi$ formed by these trajectories. Since Ψ/H is potential, we have

$$\oint_{X^*} \frac{\psi}{H}\, dX = \oint_{X^\circ} \frac{\psi}{H}\, dX$$

From the Pontryagin maximum principle it follows that the right-hand side of this relation is equal to T°, and the left-hand side does not exceed T^*, where T°, T^* is the time of motion along X° and X^* respectively to the origin of coordinates. From this it follows that $T^\circ \leqslant T^*$, i.e. X° is an OT. The theorem is proved.

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